Lattice Animals as Limits of Clusters in Percolation Theory

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We investigate if the limits $p \to 0$ or $p \to 1$ for large percolation clusters at concentration p correspond to large lattice animals for fixed size s or fixed perimeter t, respectively. A numerical analysis for the asymptotic number of lattice animals with fixed large perimeter shows that the limit $s \to \infty$ can be interchanged with the limit $p \to 0$, but the two limits $t \to \infty$ and $p \to 1$ cannot be interchanged.

In the percolation problem each site of an infinite lattice is randomly occupied with probability p; a cluster is a group of occupied sites connected by nearest-neighbor distances. Its size s is the number of occupied sites, and its perimeter t is the number of empty sites which are nearest neighbors to cluster sites. If g_{st} counts the number of possible configurations (also called "lattice animals") with size s and perimeter t, then

$$n_{st} = g_{st}\, p^s q^t\,, \quad q \equiv 1 - p \tag{1}$$

is the average number of percolation clusters of size s with perimeter t, and

$$n_s \equiv \textstyle\sum_t n_{st} \tag{2a}$$

is the total number of clusters with s sites each. Duarte [1] also investigated sums over cluster sizes s at fixed perimeter t; we call

$$n_t' \equiv \textstyle\sum_s n_{st} \tag{2b}$$

the total number of percolation clusters with perimeter t. Similarly,

$$g_s \equiv \textstyle\sum_t g_{st} \quad \text{and} \quad g_t' \equiv \sum_s g_{st} \tag{2c}$$

are the corresponding total numbers of lattice animals at fixed size s or fixed perimeter t, respectively. In Monte Carlo simulations or other numerical studies [2] one likes to treat the animal problem as a special case of the percolation problem by simply setting $p = 0$ in a computer program working at fixed size s. To what extent does the limit $p \to 0$ correspond to animals at fixed size s, and the limit $p \to 1$ to animals at fixed perimeter t, as the following Eq. (3) seems to suggest.

Clearly, if $p \to 0$ at fixed cluster size s, we have $n_{st} p^{-s} = g_{st} q^t \cong g_{st}$ from Eq. (1) or [2] after summation over all t:

$$n_s(p \to 0)\, p^{-s} = g_s\,. \tag{3a}$$

Similarly, summing up over all s for p close to unity we get

$$n_t'(p \to 1)\, q^{-t} = g_t' \tag{3b}$$

for fixed t. It is less clear that this correspondence of percolation cluster numbers n_s and n_t' to animal numbers g_s and g_t', as shown in Eq. (3), is also valid asymptotically for very large clusters, when we are interested in the limit of s (or t) going to infinity at small but fixed p (or q respectively). Equation (3), instead, is valid only for the opposite ordering of limiting processes, i.e. for p or q going to zero at large but fixed cluster size s or t. We now investigate by a cumulant expansion if these two limits are interchangeable.

The general cumulant expansion for any average denoted by $\langle \cdots \rangle$ is [3]

$$\ln \langle e^x \rangle = \langle x \rangle + \frac{1}{2}\langle (x - \langle x \rangle)^2 \rangle + \frac{1}{6}\langle (x - \langle x \rangle)^3 \rangle + \frac{1}{24}\langle (x - \langle x \rangle)^4 - 3\,(x - \langle x \rangle)^3 \rangle + \cdots.$$

Let us now look at any animal property A_{st} depending on s and t. For our purposes at $p \to 0$ we define an average by $\langle A_s \rangle = \sum_t A_{st}\, g_{st} / \sum_t g_{st}$, and at $p \to 1$ another average by $\langle A_t \rangle' = \sum_s A_{st}\, g_{st} / \sum_s g_{st}$. Then we have from (1) and (2) with $x = \ln(q^t)$ or $x = \ln(p^s)$, respectively, in the cumulant expansion:

$$\begin{aligned} \ln(n_s\, p^{-s}) &= \ln[\langle e^{t \ln q} \rangle\, g_s] \\ &= \ln(g_s) + \langle t_s \rangle \ln(q) \\ &\quad + \tfrac{1}{2}\langle (t_s - \langle t_s \rangle)^2 \rangle \ln^2(q) + \cdots, \end{aligned} \tag{4a}$$

$$\begin{aligned} \ln(n_t'\, q^{-t}) &= \ln[\langle e^{s \ln p} \rangle'\, g_t'] \\ &= \ln(g_t') + \langle s_t \rangle' \ln(p) \\ &\quad + \tfrac{1}{2}\langle (s_t - \langle s_t \rangle')^2 \rangle \ln^2(p) + \cdots. \end{aligned} \tag{4b}$$

Numerical evidence in two and three dimensions [1, 2, 4, 5] has shown that $\langle t_s \rangle = bs + \cdots$ for large s, and presumably also the width $\langle (t_s - \langle t_s \rangle)^2 \rangle$ varies as s. Moreover, for large animals,

$$\ln(g_s) = s \ln(\lambda) - \theta \ln(s) + \cdots$$

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[6]. Thus for $p \to 0$ and $s \to \infty$ (4a) gives (provided the higher terms of the cumulant expansion do not disturb us):

$$\ln(n_s\, p^{-s}) = s \ln \lambda + b\, s\, p(1 + 0(p)) + \cdots,$$

$$\text{or} \quad (n_s\, p^{-s})^{1/s} \cong \lambda\, e^{bp}\,, \tag{5}$$

where the neglected terms $(+\cdots)$ vary less than s with increasing cluster size. Thus the two limits can be interchanged for the leading terms:

$$\lim_{s\to\infty} \lim_{p\to 0} (n_s\, p^{-s})^{1/s} = \lim_{p\to 0} \lim_{s\to\infty} (n_s\, p^{-s})^{1/s} = \lambda\,. \tag{6}$$

In short, we have $n_s \sim (\lambda p)^s$ near $p = 0$ for large clusters, connecting the animal λ with the percolation n_s for small concentration, as desired.

In the other case, for p close to unity, the situation is not so nice. Duarte [1] found $\langle s_t \rangle' \propto t^y + \cdots$ for large t, with y about 3/2 significantly larger than unity, in striking contrast to the $\langle t_s \rangle \propto s$ relation mentioned above. Furthermore we analyzed the polynomials of [7] to find the numbers g_t' of animals at fixed perimeter. Assuming

$$g_t' \propto \lambda'^t\, t^{-\theta'} \quad (t \to \infty) \tag{7a}$$

we found a ratio analysis in the triangular lattice:

$$\lambda' \cong 4.6 \pm 0.3\,; \quad \theta' \cong 5.5 \pm 1\,. \tag{7b}$$

Our error bars may be regarded as rather optimistic in view of the rather strong scattering shown in Fig. 1 for the ratios g_t'/g_{t-1}' plotted versus $1/t$. Equation (7a) predicts these data to follow a straight line with intercept λ' and slope $-\theta'\lambda'$, and we see that only the last few points are approximated well by (7a). More accurately, Table 1 gives the consecutive estimates for λ' and θ', if in the ratio plot we fit a straight line through two consecutive data points. A few more points in the series of [7] would help to get more accurate extrapolations; but we feel our data show reliably that θ' is nearly one order-of-magnitude larger than the $\theta \cong 1$ exponent describing g_s. We have no explanation for this surprisingly large value of θ'.

Equation (4b) now takes for $p = 1 - q$ close to unity the form

$$\begin{aligned} \ln(n_t'\, q^{-t}) &= \ln g_t' - \langle s_t \rangle' \cdot q + \cdots \\ &= t \ln \lambda' - \text{const}\, t^y\, q + \cdots \end{aligned} \tag{8a}$$

for large clusters, and now the two limits $t \to \infty$ and $q \to 0$ cannot be interchanged:

$$\lim_{p\to 1} \lim_{t\to\infty} (n_t'\, q^{-t})^{1/t} = \infty\,;$$

$$\lim_{t\to\infty} \lim_{p\to 1} (n_t'\, q^{-t})^{1/t} = \lambda'\,. \tag{8b}$$

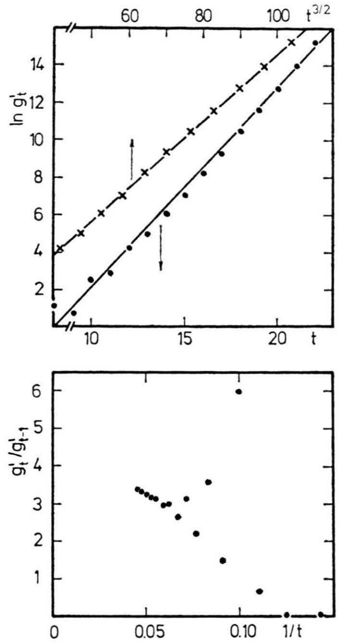

Fig. 1. Numerical test of Equation (7). The upper part shows g_t' plotted logarithmically versus t and versus t^y with $y = 3/2$. The lower part is a ratio analysis for the parameters λ' and θ', with the solid line corresponding to Equation (7b).

In this sense, while near $p = 0$ we may identify as in (3a) the percolation clusters at fixed s with the animals, $n_s \sim g_s p^s$ at least for the leading terms, this identification would be wrong near $p = 1$ at fixed t: The approximation $n_t' \sim g_t' q^t$ of (3b) does not work in the limit of t going to infinity at fixed small q. This asymmetry in the relation between large percolation clusters and large lattice animals is traced back to the large exponent y observed [1] in the size-versus-perimeter relation of animals.

[One might speculate that instead of (7a) a different asymptotic form like $g_t' \propto t^{-\theta''} \lambda''^{(t^y)}$ is valid. Indeed Fig. 1 shows this form to fit our data

Table 1. Consecutive estimates of parameters λ' and θ' found from two consecutive ratios g_t'/g_{t-1}' of Figure 1. ($\lambda' = \varrho_t + (t-1)^2(\varrho_t - \varrho_{t-1})/t$ and $\theta' = (\varrho_t - \varrho_{t-1})(t-1)^2/\lambda'$ with $\varrho_t = g_t'/g_{t-1}'$.) Only the last points, giving the best extrapolation, are shown.

t	ϱ_t	λ_t'	θ_t'
18	3.16	5.85	8.28
19	3.18	3.60	2.19
20	3.28	4.95	6.76
21	3.33	4.40	5.08
22	3.40	4.78	6.35

quite well. But then Eq. (8a) would be replaced by $\ln(n_t' q^{-t}) = t^y \ln(\lambda'') - \text{const}\, t^y q + \cdots$, making the limits $q \to 0$, $t \to \infty$ interchangeable now. Then we would have for $q \to 0$ and $t \to \infty$: $n_t' \sim q^t \lambda''(t^y)$, with both $y \cong 3/2$ and λ'' larger than unity. Thus for $t \to \infty$ at fixed small q the cluster numbers n_t' would go to infinity. But the sum $\sum_t t n_t'$ is the probability that an arbitrary lattice site belongs to the perimeter of some finite cluster; this probability cannot be larger than unity, and thus n_t' must go to zero, not to infinity, for $t \to \infty$. This contradiction shows that $\log(g_t')$ cannot vary more quickly than t with increasing t, and in particular cannot vary as $t^{3/2}$. Therefore (7a) presumably is correct.]

In conclusion we found that the asymptotic behavior of large clusters is more complicated for $p \to 1$ than for $p \to 0$. Thus, for example, nucleation theory [8], which needs cluster numbers for $p \to 1$, presumably cannot use for these cluster numbers the exponent θ' determined in (7), since that exponent refers to a limiting process ($p = 1$) different from the one needed in nucleation theory ($p \to 1$).

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